Rational Homotopy Groups of Diffeomorphisms of Discs

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The goal is to prove, or understand the proof of the following theorem

$$\pi_*(\mathrm{BDiff}_\partial(D^n)) \otimes \mathbb{Q} \cong \begin{cases} 0, & n \text{ even} \\ K_{*+1}(\mathbb{Z}) \otimes \mathbb{Q}, & n \text{ odd} \end{cases}$$

when $* \leq \min\left(\frac{n-1}{3}, \frac{n-5}{2}\right)$, the so called stable range. In particular we want to look out for the areas where we might improve to integral results.

1 Sketch

First we need to define and understand the following spaces $A(X), \mathrm{Wh}^{\mathrm{Diff}}(X), QX_+, H(X), \mathcal{H}(X), \mathrm{Diff}^b_\partial(D^n)$ and $\mathcal{C}(X)$.

First the fiber sequence

$$\mathrm{Diff}_\partial(D^n) \to \mathrm{Diff}^b_\partial(D^n) \to \mathrm{Diff}^b_\partial(D^n)/\mathrm{Diff}_\partial(D^n)$$

induces a rational equivalence $\mathrm{Diff}_\partial(D^n) \simeq_{\mathbb{Q}} \mathrm{Diff}^b_\partial(D^n)/\mathrm{Diff}_\partial(D^n)$. There is a spectral sequence which starts at $E^1_{pq} = \pi_q(\mathcal{C}(M \times I^p))$ which converges to the homotopy groups of $\mathrm{Diff}^b_\partial(D^n)/\mathrm{Diff}_\partial(D^n)$, thus if we can compute the psudo-isotopy groups then using this spectral sequence we can get the homotopy groups we want.

If we know what these things are we will proceed as follows. We have a fiber sequence

$$\mathrm{BDiff}_\partial(D^{n+1}) \to \mathcal{C}(D^n) \to \mathrm{BDiff}_\partial(D^n)$$

which through spectral sequence techniques will allow us to compute the rational homotopy groups of the discs if we knew them for the concordance groups.

The moduli space of h-cobordisms $H(D^n)$ splits as $H(D^n) \simeq \mathrm{Wh}_1(\pi_1(D^n)) \times \mathrm{B}\mathcal{C}(D^n)$ where Wh_1 is, using $\pi_1(D^n) = 1$, by definition $\mathrm{Wh}_1(1) = K_1(\mathbb{Z}[1])/(\pm 1) = K_1(\mathbb{Z})/\pm 1 = \mathbb{F}_2/\pm 1 = 1$ the trivial group, thus there is a weak equivalence between $B\mathcal{C}(M) \simeq H(M)$. Igusa shows that in a range we have that $H(M) \simeq \mathcal{H}(M)$ the space of stable h-cobordisms. Therefore we have shown that to compute the rational diffeomorphisms of discs, at least in a range, we just need to compute the rational homotopy groups of the stable h-cobordism group. Now Waldhausen showed that

$$\mathcal{H}(M) \simeq \Omega \mathrm{Wh}^{\mathrm{Diff}}(M)$$

Thus it reduces to computing the rational groups of the smooth whitehead spectrum. Finally this can be computed for a disc because we have that

$$A(X) \simeq \mathrm{Wh}^{\mathrm{Diff}}(X) \times QX_+$$

which is has been done for the disc.

1.1 Integrally

Notice that we have weak equivalences on the nose (in a range) for

$$\mathrm{B}\mathcal{C}(D^n) \simeq H(D^n) \simeq_{i \leq I} \mathcal{H}(D^n) \simeq \Omega \mathrm{Wh}^{\mathrm{Diff}}(D^n)$$

which rationally extends to

$$\mathrm{B}\mathcal{C}(D^n) \simeq H(D^n) \simeq_{n \leq I} \mathcal{H}(D^n) \simeq \Omega \mathrm{Wh}^{\mathrm{Diff}}(D^n) \simeq_{\mathbb{Q}, i \neq 0} A(*) \simeq_{\mathbb{Q}} K(\mathbb{Z})$$

Thus by a careful analysis of the smooth whitehead spectrum of the disc we might hope to get some integral results. To make this tighter I need to understand the precise integral relationship between the A theory of a point and the K theory of the integers (is there one), and what the integral relation between the Whitehead spectrum and the A theory of the point is.

Finally to get from the concordance group to the disc we need to inspect the spectral sequence argument and see how much of thta relies on rational hypotheses.

2 Diffeomorphisms to Block Diffeomorphism

$$\mathrm{Diff}_\partial(D^n) \simeq_{\mathbb{Q}} \mathrm{Diff}^b_\partial(D^n)/\mathrm{Diff}_\partial(D^n) \Leftarrow \mathrm{B}\mathcal{C}(D^n) \simeq H(D^n) \simeq_{n \leq I} \mathcal{H}(D^n) \simeq \Omega \mathrm{Wh}^{\mathrm{Diff}}(D^n) \simeq_{\mathbb{Q}, i \neq 0} A(*) \simeq_{\mathbb{Q}} K(\mathbb{Z})$$

The group of diffeomorphisms acts on the group of block diffeomorphisms. If we beleive that this group action is transitive then it will have trivial stabilisers. Now the quotient $\mathrm{Diff}^b_\partial(D^n)/\mathrm{Diff}_\partial(D^n)$ has elements that are given by cosets, $f\mathrm{Diff}_\partial(D^n)$ where $f \in \mathrm{Diff}^b_\partial(D^n)$, which are clearly in bijection with $\mathrm{Diff}_\partial(D^n)$ for any f. Thus at least on the level of sets we have a surjection

$$\mathrm{Diff}^b_\partial(D^n) \to \mathrm{Diff}^b_\partial(D^n)/\mathrm{Diff}_\partial(D^n)$$

and the preimage of any point in the quotient is in bijection with the required fiber.

The final point is to see that the block diffeomorphism group vanishes rationally.

3 The Spectral Sequence

$$\mathrm{Diff}_\partial(D^n) \simeq_{\mathbb{Q}} \mathrm{Diff}^b_\partial(D^n)/\mathrm{Diff}_\partial(D^n) \Leftarrow \mathrm{B}\mathcal{C}(D^n) \simeq H(D^n) \simeq_{n\leq I} \mathcal{H}(D^n) \simeq \Omega \mathrm{Wh}^{\mathrm{Diff}}(D^n) \simeq_{\mathbb{Q}, i\neq 0} A(*) \simeq_{\mathbb{Q}} K(\mathbb{Z})$$

We start with a couple of definitions. Let $D^k(M)$ be the quotient of simplicial groups (is it with respect to k or something else), the diffeomorphisms of $M \times I^k$ rel the boundary that preserve the projection to I^k on $M \times \partial I^k$ modulo those that always preserve the projection.

// image

$C^k(M)$ is similarly the diffeos of $M \times I^k \times I$ rel boundary that preserver the projection $I^k \times I$ on $M \times I^k \times \{0\} \cup M \times \partial I^k \times I$.

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Then there is a fibration

$$D^{k+1}(M) \to C^k(M) \to D^k(M)$$

then taking the LES in homotopy groups we get a sequece that is exact except at the final arrow

$$\cdots \to \pi_1(D^k) \to \pi_0(D^{k+1}) \to \pi_0(C^k) \to \pi_0(D^k) \to 0$$

this can be made exact in the negative places, and then rolled up into an exact couple

$$\begin{tikzcd}
\begin{cases} \pi_i D^k(M), & i \geq 0 \\ \pi_{i+k}\mathrm{Diff}^b_\partial(M)/\mathrm{Diff}_\partial(M), & \text{else} \end{cases} = D^{i,k} \arrow[rr] & & D^{i-1,k+1} \arrow[dl] \\
& \begin{cases} \pi_i C^k(M), & i \geq 0 \\ 0, & \text{else} \end{cases} = E^{i-1,k} \arrow[ul] &
\end{tikzcd}$$

Summing over the k as well gives us an exact couple of bimodules that form the E^1 page of a homologically graded spectral sequence. It is a theorem that this SS is first quadrant and therefore converges, moreover it converges to

$$E^1_{p,q}(M) \implies \pi_{p+q+1}\mathrm{Diff}^b_\partial(M)/\mathrm{Diff}_\partial(M)$$

To understand the differential we first note that $C^k(M) \simeq C(M \times I^k)$, thus connecting us to the concordance spaces that we have previously computed. We will substitute these as the groups on the E_1 page. There is then a pair of maps that will help us describe the differential

$$\begin{tikzcd}
C(M \times I^{k-1}) \arrow[r, bend left, "\mathrm{res}|_{M\times I^{k-1}\times\{1\}}"] & C(M \times I^k) \arrow[l, bend left, "\sigma = - \times \mathrm{id}_I"]
\end{tikzcd}$$

the bottom map is sometimes refered to as the stabilisation map because for $k << \dim(M \times I^k)$ it is k connected, that is a surjection on the $k-th$ homotopy group and an iso on the $i < k$ homotopy groups. The maps induced on homotopy groups we will denote $\pi_*(\mathrm{res}), \sigma_*$ respectively. The differential is given by $d_1 = \pi_*(\mathrm{res})$. The other map will allow us to express this differential in a simple form in the stable range.

Given an element $g \in C(N)$ then we can "reverse" it (intuitively reversing the direction of the h-cobordism) by applying

$$g \mapsto \bar{g} \quad := \quad \left(g|_{N\times\{1\}} \times \mathrm{id}_I\right)^{-1} \circ g \circ \left(\mathrm{id}_N \times \tau\right)$$

where $\tau : I \to I, t \mapsto 1-t$. This defines an involution. Now we can relation the stabilisation map to this involution by two formulas:

$$d_1\sigma_*[g] = [g] + [\bar{g}]$$
$$\overline{[\sigma(g)]} = -\sigma_*[\bar{g}]$$

Note that our σ is surjective in a range [Hat78], $k \leq (\dim M - 10)/6$, although this bound has likely been improved and so this first formula in a range can always be used.

3.0.1 Applying it to the disc

With this setup in mind we will start to compute the spectral sequence in the case that $M = D^n$. First notice that we are starting on E^1 and are trying to deduce the E^∞ page. Using the fact that we can smooth corners, i.e. $\pi_i C(D^n \times I^k) \cong \pi_i C(D^{n+k})$, the E^1 page is given by

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \pi_2 C(D^n) & \longleftarrow & \pi_2 C(D^{n+1}) & \longleftarrow & \pi_2 C(D^{n+2}) & \longleftarrow & \cdots \\
 & & & & & & & & \\
0 & \longleftarrow & \pi_1 C(D^n) & \longleftarrow & \pi_1 C(D^{n+1}) & \longleftarrow & \pi_1 C(D^{n+2}) & \longleftarrow & \cdots \\
 & & & & & & & & \\
0 & \longleftarrow & \pi_0 C(D^n) & \longleftarrow & \pi_0 C(D^{n+1}) & \longleftarrow & \pi_0 C(D^{n+2}) & \longleftarrow & \cdots
\end{array}$$

The punch line of the theorems below is the following

$$\begin{aligned}
\pi_* C(D^{n+k})_{\mathbb{Q}} &\cong \pi_{*+2}\mathrm{Wh}^{\mathrm{Diff}}(D^{n+k})_{\mathbb{Q}} \\
&\cong \pi_{*+2}\mathrm{Wh}^{\mathrm{Diff}}(*)_{\mathbb{Q}} \\
&\cong \begin{cases} 0, & *+2=0 \\ K_{*+2}(\mathbb{Z})_{\mathbb{Q}}, & \text{else} \end{cases} \\
&\cong \begin{cases} 0, & \text{else} \\ \mathbb{Q}, & * = -2 \\ \mathbb{Q}, & * \equiv 3 \pmod 4, \text{ and } * \geq 3 \end{cases} \\
&\cong \begin{cases} 0, & \text{else} \\ \mathbb{Q}, & * \equiv 3 \pmod 4 \end{cases}
\end{aligned}$$

where we have used that $* \geq 0$. Notice that this doesnt depend on $n+k$ and so the groups on each row will all be the same, we get every fourth row starting at row three being $\mathbb{Q}$'s and the rest are zeroes. Note that the $\mathbb{Q}$'s should be interpreted as the K theory of the integers, as this is how one computes the differentials. Subbing in these values we get

$$\begin{array}{ccccccc}
11 & \mathbb{Q} & \longleftarrow & \mathbb{Q} & \longleftarrow & \mathbb{Q} & \longleftarrow \cdots \\
\\
7 & \mathbb{Q} & \longleftarrow & \mathbb{Q} & \longleftarrow & \mathbb{Q} & \longleftarrow \cdots \\
\\
3 & \mathbb{Q} & \longleftarrow & \mathbb{Q} & \longleftarrow & \mathbb{Q} & \longleftarrow \cdots \\
\\
q/p & 0 & & 1 & & 2 &
\end{array}$$

where everything else is zero. Now $K_i(\mathbb{Z})_{\mathbb{Q}}$ has its own involution induced by the transpose inverse on $\mathrm{GL}_i(\mathbb{Z})$. This involution was computed by Farrell-Hsiang as given by multiplication by -1 for $i \neq 0$ and multiplication by 1 for $i = 0$. The final fact is that we have a commuting diagram

$$\begin{tikzcd}
\mathbb{Q} = \pi_{i-2}C(D^n) \arrow[r, "\mathrm{id}"] \arrow[d, "\overline{(\,)}"] & \mathbb{Q} = K_i(\mathbb{Z})_{\mathbb{Q}} \arrow[d, "\times(-1)"] \\
\mathbb{Q} = \pi_{i-2}C(D^n) & \mathbb{Q} = K_i(\mathbb{Z})_{\mathbb{Q}} \arrow[l, "\times(-1)^{n+1}"]
\end{tikzcd}$$

which we can now use with our previous lemmas to compute the differentials in the stable range

$$d_1(\sigma_*[f]) = [f] + [\bar{f}] = [f] + (-1)(-1)^{n+1}[f] = \begin{cases} 2[f], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

Thus the groups dont depend on the horrozontal axis but the maps do. Notice also that multiplication by two is a *rational* isomorphism. Thus depending on whether n is odd or even the first map will either be zero or an isomorphism and then it alternates from there:

$$n \text{ even}: \mathbb{Q} \xleftarrow{\simeq} \mathbb{Q} \xleftarrow{0} \mathbb{Q} \xleftarrow{\simeq} \mathbb{Q} \xleftarrow{0} \mathbb{Q} \leftarrow \cdots$$

$$n \text{ odd}: \mathbb{Q} \xleftarrow{0} \mathbb{Q} \xleftarrow{\simeq} \mathbb{Q} \xleftarrow{0} \mathbb{Q} \xleftarrow{\simeq} \mathbb{Q} \leftarrow \cdots$$

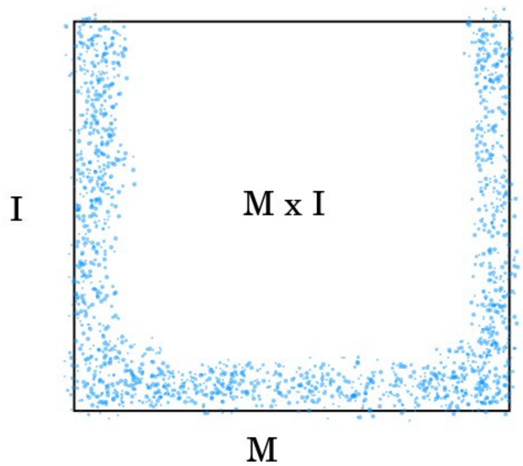
In the even case clearly all the pairs cancel out in homology, in the odd case all the pairs *except the first* $\mathbb{Q}$ cancel, so we are left with only $\mathbb{Q}$'s in the first column for n odd, and it is clear that the sequence will have stabilised as the maps will never again go between two non-zero groups.

Remark. The involution defines a $\mathbb{Z}_2$ action on the E^1 page by $-1.[f] \mapsto (-1)^k[\bar{f}]$ between $\pi_*C(M \times I^k)$. This action allows one to show that the E^2 page has a nice expression in terms of the group homology of $\mathbb{Z}_2$.

4 Moduli of h-cobordisms

$$\mathrm{Diff}_\partial(D^n) \simeq_{\mathbb{Q}} \mathrm{Diff}^b_\partial(D^n)/\mathrm{Diff}_\partial(D^n) \Leftarrow \mathrm{B}\mathcal{C}(D^n) \simeq H(D^n) \simeq_{n \leq I} \mathcal{H}(D^n) \simeq \Omega\mathrm{Wh}^{\mathrm{Diff}}(D^n) \simeq_{\mathbb{Q}, i \neq 0} A(*) \simeq_{\mathbb{Q}} K(\mathbb{Z})$$

First of all we define our terms. $\mathcal{C}(M) = \mathrm{Diff}(M \times I \text{ rel } M \times \{0\} \cup \partial M \times I)$ is the group of concordances, identified up to psuedo-isotopy. In pictures it is the diffeomorphisms of $M \times I$ that fix the shaded region in 4. The moduli of h-cobordisms of M is given by $H(M)$, this is a topological space



whose points are compact manifolds with a boundry with two components, one of which is M such that there is a deformation retract onto both of the boundary components. Such h-cobordisms are identified up to diffeomorphism *fixing* M. In other words its elements are just h-cobordisms between M and another space. There are many suitable ways to topologise this, [Kup] gives an explicit simplicial set.

Because the disc is simply connected we may apply the h-cobordism theorem to see that any h-cobordism between the disc and something else is just a cylinder (diffeomorphically), that is $D^n \times I$. Hence we have that all the elements of $H(D^n)$ are up to diffeomorphism just $D^n \times I$. One subtilty is that we are identifing cobordisms up to diffeomorphisms that fix M, thus we might write (or define) that $H(D^n) = \mathrm{Emb}_M(D^n \times I, \mathbb{R}^\infty)$, where the subscript M denotes maps that fix M, and we take these maps up to psudo-isotopy.

By the standard trick of "pulling out the middle" of the square in 4, and retracting the sides we see that $\mathcal{C}(M) \cong \mathrm{Diff}(M \times I \text{ rel } M \times \{0\}) = \mathrm{Diff}_M(M \times I)$, diffeomorphisms that fix one of the ends. It is then clear (an embedding is a diffeo onto its image, if we fix an embedding then we can compose with the diffeomorphisms to get all the other embeddings, this is like claiming that the action of the diff group on embeddings is free and transitve and therefore it is a bijection by the orbit stabiliser) that

$$\mathrm{Diff}_M(M \times I) \cong \mathrm{Emb}_M(M \times I, \mathbb{R}^\infty)$$

where both sides are up to psudo-isotopy.

5 Theorem of Igusa

$$\mathrm{Diff}_\partial(D^n) \simeq_{\mathbb{Q}} \mathrm{Diff}^b_\partial(D^n)/\mathrm{Diff}_\partial(D^n) \Leftarrow \mathrm{B}\mathcal{C}(D^n) \simeq H(D^n) \simeq_{n \leq I} \mathcal{H}(D^n) \simeq \Omega \mathrm{Wh}^{\mathrm{Diff}}(D^n) \simeq_{\mathbb{Q}, i \neq 0} A(*) \simeq_{\mathbb{Q}} K(\mathbb{Z})$$

Igusas theorem is that the suspension map

$$\sigma : C^{\mathrm{Diff}}(M) \to C^{\mathrm{Diff}}(M \times J)$$

is k connected (isomorphism on $i \leq k$ homotopy groups, surjective on others) for all compact smooth n-manifolds with $n \geq \max\{2k+1, 3k+4\}$. [Igu88] has a section in the introduction summarising the proof (for only a disc) which we will crib here.

The first step is to define what the suspension map is, for the case of the disc we will assume freely that we can round corners. Then the suspension map is

$$\sigma : \mathrm{Diff}(D^n) \to \mathrm{Diff}(D^{n+1})$$

given by $f \mapsto f \times \mathrm{id}$. In this case we are using that $D^n \times I \cong D^{n+1}$. The concordance group (of smooth psuedo-isotopies) for D^n is a subset of $\mathrm{Diff}(D^{n+1})$ and as such this clearly defines a sequence

$$\mathcal{C}(D^n) \xrightarrow{\sigma} \mathcal{C}(D^n \times I) \to \mathcal{C}(D^n \times I^2) \to \cdots$$

We want to show that the first map in the sequence is $\max\big\{(n-4)/3, (n-7)/2\big\}$ connected (the bounds can be improved), we will go by a stable result first. So define the limit of the above sequence as $\mathcal{P}(D^n) = \mathrm{colim}_k \mathcal{C}(D^n \times I^k)$. The stable result states that the induced map $\mathcal{C}(D^n) \to \mathcal{P}(D^n)$ is a (split) epimorphism on π_k where $k \leq (n-9)/2$.

To show the stable result we begin by constructing a classifying space. Hatcher showed that

$$\mathrm{Diff}(D^n) \simeq \mathrm{O}(n) \times \mathcal{C}(D^{n-1})$$

There is a free action $\mathrm{Diff}(D^n) \curvearrowright \mathrm{Emb}(D^n, \mathbb{R}^n)$. The orbit space is the space of submanifolds of $\mathbb{R}^n$ diffeomorphic to D^n. By a lemma of Palise-Cerf we have that

$$\mathrm{Emb}(D^n, \mathbb{R}^n) \simeq \mathrm{O}(n)$$

And hence submanifolds of $\mathbb{R}^n$ diffeomorphic to D^n is a model for the concordance group $\mathcal{C}(D^{n-1})$

$$\mathrm{Emb}(D^n, \mathbb{R}^n)/\mathrm{Diff}(D^n) \simeq \mathrm{O}(n)/\mathrm{Diff}(D^n) \simeq \mathrm{Diff}(D^n)/\mathrm{O}(n) \simeq \mathcal{C}(D^{n-1})$$

Then using this the proof goes by using Cerf theory ("half Morse theory") and Hatchers two index theorem.

[WJR] states that from the above theorem we get the equivalence required by delooping (suspend or B?) once and iterating it follows that the infinite stabilization map $H^{Diff}(M) \to \mathcal{H}^{Diff}(M)$ is $k+1$ connected for k, n as above.

6 Theorem of Waldhausen

$$\mathrm{Diff}_\partial(D^n) \simeq_{\mathbb{Q}} \mathrm{Diff}^b_\partial(D^n)/\mathrm{Diff}_\partial(D^n) \Leftarrow \mathrm{B}\mathcal{C}(D^n) \simeq H(D^n) \simeq_{n \leq I} \mathcal{H}(D^n) \simeq \Omega \mathrm{Wh}^{\mathrm{Diff}}(D^n) \simeq_{\mathbb{Q}, i \neq 0} A(*) \simeq_{\mathbb{Q}} K(\mathbb{Z})$$

This theorem is the content of [WJR]. We start with the definitional fiber sequence for PL manifolds (PL and Top are the same in this context) given by

$$h(M; A(*)) \to A(M) \to \mathrm{Wh}^{PL}(M)$$

where $h(M; A(*))$ is the cohomology with coefficients in the spectrum $A(*)$. They then write down the following large diagram:

$$\begin{tikzcd}
\mathcal{H}^{PL}(M) \arrow[r] \arrow[d] & \mathrm{colim}_n \mathcal{M}^n_\bullet \arrow[r] \arrow[d] & \mathrm{colim}_n h\mathcal{M}^n_\bullet \arrow[d] \\
s\tilde{\mathcal{E}}^h_\bullet(M) \arrow[r] & s\tilde{\mathcal{E}}_\bullet \arrow[r] & h\tilde{\mathcal{E}}_\bullet \\
s\mathcal{D}^h(X) \arrow[r] \arrow[u] \arrow[d] & s\mathcal{D} \arrow[r] \arrow[u] \arrow[d] & h\mathcal{D} \arrow[u] \arrow[d] \\
s\mathcal{C}^h(X) \arrow[r] & s\mathcal{C} \arrow[r] & h\mathcal{C}
\end{tikzcd}$$

And the proof then proceeds by showing that the horizontal rows are homotopy fiber sequences, and the middle and right columns are all homotopy equivalences. Therefore the fibers of the maps are also homotopy equivalences. Then a theorem of Waldhausen implies that there is a homotopy equivalence

$$|s\mathcal{C}^h(X)| \simeq \Omega \mathrm{Wh}^{PL}(X).$$

The moral of each of these equivalences is going from the first row to the second is viewing stably framed PL manifolds as polyhedra and their bundles, $\mathcal{M}$, as PL Serre fibrations, $\tilde{\mathcal{E}}$. From the second to the third row is vewing finite non-singular simplicial complexes, $\mathcal{D}$,as polyhedra via the geometric realisation. The final equivalence is vewing finite non-singular simplicial sets as finite general simplicial sets, $\mathcal{C}$.

Now from the PL case we deduce the smooth case, again using the defining fiber sequence for the smooht White head spectrum

$$Q(M_+) \to A(M) \to \mathrm{Wh}^{\mathrm{Diff}}(M)$$

The argument to go from the PL to Diff case is outlined on page (15). They use homotopy functors and homotopy fibers of a bunch of maps to deduce some weak equivalences. Smoothing theory (Morlets disjunction lemma) implies that some PL group is contractable. Then things are compared with stabilisations.

Remark. Actually they show something stronger that not only is there an equivalence but that there is a homotopy fiber sequence

$$\mathcal{H}^{PL}(M) \to h(M; A(*) \to A(M)$$

which follows from the homotopy equivalence above and the definitional homotopy sequence of the Whitehead space, but we looped the space and so its place in the homotopy LES will move up one?

7 Homotopy groups of the smooth Whitehead spectrum

$$\mathrm{Diff}_\partial(D^n) \simeq_{\mathbb{Q}} \mathrm{Diff}^b_\partial(D^n)/\mathrm{Diff}_\partial(D^n) \Leftarrow \mathrm{B}\mathcal{C}(D^n) \simeq H(D^n) \simeq_{n \le I} \mathcal{H}(D^n) \simeq \Omega \mathrm{Wh}^{\mathrm{Diff}}(D^n) \simeq_{\mathbb{Q}, i \neq 0} A(*) \simeq_{\mathbb{Q}} K(\mathbb{Z})$$

Both the Waldhausen A theory and the Whitehead spectrum are weak homotopy invariants on path connected spaces and therefore it is clear that

$$A(D^n) = A(*), \quad \mathrm{Wh}^{\mathrm{Diff}}(D^n) = \mathrm{Wh}^{\mathrm{Diff}}(*)$$

Moreover we have that

$$QD^n_+ = \Omega^\infty \Sigma^\infty (D^n \sqcup *) = \Omega^\infty \mathbb{S}$$

the sphere spectrum. At the level of spectra we have the decomposition

$$A(*) = \mathbb{S} \times \mathrm{Wh}^{\mathrm{Diff}}(*)$$

The homotopy groups are taking a colimit and colimits preserve products (in a Cartesian closed category such as spectra) so we have that

$$\pi_* A(*) = \pi_* \mathbb{S} \oplus \pi_* \mathrm{Wh}^{\mathrm{Diff}}(*)$$

but the homotopy groups of the sphere spectrum are just the stable homotopy groups of spheres which are all finite (therefore torsion) except in degree 0 where it is $\mathbb{Z}$ and so rationally

$$\pi_* A(*) \otimes \mathbb{Q} = \begin{cases} (\mathbb{Z} \oplus \pi_0 \mathrm{Wh}^{\mathrm{Diff}}(*)) \otimes \mathbb{Q}, & * = 0 \\ \pi_* \mathrm{Wh}^{\mathrm{Diff}}(*) \otimes \mathbb{Q}, & \text{else} \end{cases}$$

Do these spectra have lower degrees? It is known however that $\pi_0 A(*) = \mathbb{Q}$ and therefore we have that $(\mathbb{Z} \oplus \pi_0 \mathrm{Wh}^{\mathrm{Diff}}(*)) \otimes \mathbb{Q} = (\mathbb{Z} \otimes \mathbb{Q}) \oplus (\pi_0 \mathrm{Wh}^{\mathrm{Diff}}(*) \otimes \mathbb{Q}) = \mathbb{Q}$ hence $\pi_0 \mathrm{Wh}^{\mathrm{Diff}}(*) \otimes \mathbb{Q} = 0$. Thus what is left is to compute $\pi_* A(*) \otimes \mathbb{Q}$ for $* \neq 0$.

8 A theory of a point

$$\mathrm{Diff}_\partial(D^n) \simeq_{\mathbb{Q}} \mathrm{Diff}^b_\partial(D^n)/\mathrm{Diff}_\partial(D^n) \Leftarrow \mathrm{B}\mathcal{C}(D^n) \simeq H(D^n) \simeq_{n \leq I} \mathcal{H}(D^n) \simeq \Omega \mathrm{Wh}^{\mathrm{Diff}}(D^n) \simeq_{\mathbb{Q}, i \neq 0} A(*) \simeq_{\mathbb{Q}} K(\mathbb{Z})$$

Kupers gives us a weak equivalence proof and of what?

$$\mathrm{End}_{Sp}(\vee_n \mathbb{S}) \simeq \prod_n \Omega^\infty(\mathbb{S})^n$$

We can compute the π_0 of the right hand side (and therefore the left) as follows. First π_0 commutes with products so it suffices to compute $\pi_0 \Omega^\infty(\mathbb{S})^n$, which we can do by appling adjunctions and the fact that functions into products are products of functions

$$\begin{aligned} \pi_0 \Omega^\infty(\mathbb{S})^n &= [S^0, \Omega^\infty(\mathbb{S})^n] \\ &= [\Sigma^\infty S^0, \mathbb{S}^n] \\ &= [\Sigma^\infty S^0, \Sigma^\infty S^0]^n \\ &= [S^0, S^0]^n_{\text{stable}} \\ &= \left(\pi_0^S\right)^n \\ &= \mathbb{Z}^n \end{aligned}$$

and therefore we have that

$$\pi_0 \mathrm{End}_{Sp}(\vee_n \mathbb{S}) = \pi_0 \prod_n \Omega^\infty(\mathbb{S})^n = \prod_n \pi_0 \Omega^\infty(\mathbb{S})^n = \prod_n \mathbb{Z}^n = \mathbb{Z}^{n^2}$$

which is isomorphic to $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z})$. This group of endomorphisms has a subgroup $\mathrm{GL}_n(\mathbb{Z})$. Therefore we define a subspace of $\mathrm{End}_{Sp}(\vee_n \mathbb{S})$ by those functions that land in $\mathrm{GL}_n(\mathbb{Z})$ under π_0 and call it $\mathrm{GL}_n(\mathbb{S})$ Not obvious that this is well defined, im using a lot of isomorphisms and a functor. There is therefore by definition a surjection

$$M_n(QS^0)_{\mathrm{id}} \to \mathrm{GL}_n(\mathbb{S}) \to \mathrm{GL}_n(\mathbb{Z})$$

where we define $M_n(QS^0)_{\mathrm{id}}$ to be its fiber over the identity in $\mathrm{GL}_n(\mathbb{Z})$. Here we have that $QS^0 = \Omega^\infty \mathbb{S}$. Now the components of QS^0 are weakly equivalent and therefore $M_n(QS^0)_{\mathrm{id}}$ is weakly equivalent to

$$M_n(QS^0)_{\mathrm{id}} \simeq \left(QS^0\right)^{n^2}$$

But since $QS^0 = \Omega^\infty \mathbb{S}$ its homotopy groups are the stable homotopy groups of S^0 or the stable stems, that is rationally zero in non-zero degree, as they are finite. Thus Except in degree zero the fiber is rationally weakly contractable. Applying the LES in homotopy groups (rationally) therefore gives an rational weak equivalence away from degree zero

$$\mathrm{GL}_n(\mathbb{S}) \to \mathrm{GL}_n(\mathbb{Z})$$

If we take the disjoint union and the ΩB over $n \geq 0$ then we have a weak rational equivalence between

$$\Omega B\big(\sqcup_{n\geq 0} B\,\mathrm{GL}_n(\mathbb{S})\big) \to \Omega B\big(\sqcup_{n\geq 0} B\,\mathrm{GL}_n(\mathbb{Z})\big)$$

The RHS is the Quillen construction of $K(\mathbb{Z})$ and the LHS is by a theorem of Waldhausen the A-theory of the point. What is happeining in degree zero? Is this proof correct?

Remark. (Christian) The following was the "modern" perspective on this statement that was described to me by Christian. First to take the K theory of a ring we would usually start with the category of finitely generated projective modules over that ring. Instead we will "derive" this situation. Consider the category of perfect complexes over a ring, that is bounded chain complexes of finitely generated free R modules. This category inherits the Waldhausen structure from the full category of modules, with cofibrations just being degreewise cofibrations and weak equivalences are qism of chains. This gives an equivalence of the K groups

$$K(\text{f.g proj modules / R}) \to K(\text{perfect R complexes})$$

given by including the module at degree zero (and then this induces a map on K theory). There is also a clear map from the category of retractive spaces over a finite CW complex to the category of perfect complexes over $\mathbb{Z}$, given by Sing_*, that is taking the singular chain. Note that these chains will only be perfect *up to quasi isomorphism*. This is a functor of Waldhausen categories and so again induces a map on K theory

$$A(*) \to K(\mathbb{Z}).$$

The next thing to change perspective on is the LHS of this equation. In particular given a nice space X there is an equivalence of categories

$$\text{finite retractive / X} \to \mathrm{perf}\big(\mathbb{S}[\Omega X]\big)$$

given by applying Σ^∞. The RHS is perfect modules over the ring spectrum $\mathbb{S}[\Omega X]\big)$. Given a ring there is an associated ring spectrum which is specified by $\pi_i^{st}(HR) = \delta_{i=0}R$. For any such ring spectrum there is a unit map

$$\epsilon : \mathbb{S} \to HR$$

and in the case of the integers this is a rational equivalence. The final step is to then compare these new things, which is non-trivial but apparently possible using some newer results.

Remark. This deriving can maybe be thought of as something like Atiyah-Bott-Shapiro, in that the chain is now a sequence of vector bundles.

Remark. A ring spectrum is the following thing: It is an object in the monoidal category of symmetric spectra that satisfyes the ring object diagrams only up to coherent homotopy. Notice that we are not descending to the homotopy category. Such a thing is called an A_∞-ring spectrum.

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